vertical field of force, F, say, by a globe of radius a with its centre at height c and earthed. The globe acquires such a charge e as reduces its potential to zero: thus $-\mathrm{F}c+e/a$ must vanish, so that its charge is $\mathrm{F}ca$; the potential of the field therefore becomes

$$\mathrm{V} = -\mathrm{F}z+\mathrm{F}ca\left(\frac{1}{r_1}-\frac{1}{r_2}\right),$$

where r_1, r_2 represent distances from the centres of the globe and its image. If the lines of equal potential which pass underneath the globe are continued down the length of the rod to meet those coming in from the side, the circumstances of the rod with conducting region at its summit will be roughly represented.

The diagram 1 represents a uniform field with vertical lines of force and horizontal equipotential lines overhead, as disturbed by a thin conductor of the form of a semi-ellipsoid standing on the ground; the left-hand half of the field is drawn.

The diagram 2 represents a similar field overhead disturbed by the presence of a conducting sphere connected to earth by an infinitely thin wire.

Some Calculations in Illustration of Fourier's Theorem.

By Lord Rayleigh, O.M., F.R.S.

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According to Fourier's theorem a curve whose ordinate is arbitrary over the whole range of abscissæ from $x=-\infty$ to $x=+\infty$ can be compounded of harmonic curves of various wave-lengths. If the original curve contain a discontinuity, infinitely small wave-lengths must be included, but if the discontinuity be eased off, infinitely small wave-lengths may not be necessary. In order to illustrate this question I commenced several years ago calculations relating to a very simple case. These I have recently resumed, and although the results include no novelty of principle they may be worth putting upon record.

The case is that where the ordinate is constant (π) between the limits ± 1 for x and outside those limits vanishes.

In general

$$\phi(x) = \frac{1}{\pi}\int_0^{\infty} dk \int_{-\infty}^{+\infty} dv\, \phi(v) \cos k(v-x). \tag{1}$$

Here

$$\int_{-\infty}^{+\infty} dv\,\phi(v)\cos k(v-x) = 2\pi\cos kx\int_0^1 dv\cos kv = 2\pi\cos kx\,\frac{\sin k}{k}$$

$$= \frac{\pi}{k}\{\sin k(x+1)-\sin k(x-1)\},$$

and

$$\phi(x) = \int_0^{\infty}\frac{dk}{k}\{\sin k(x+1)-\sin k(x-1)\}. \tag{2}$$

As is well known, each of the integrals in (2) is equal to $\pm\frac{1}{2}\pi$; so that, as was required, $\phi(x)$ vanishes outside the limits ± 1 and between those limits takes the value π. It is proposed to consider what values are assumed by $\phi(x)$ when in (2) we omit that part of the range of integration in which k exceeds a finite value k_1.

The integrals in (2) are at once expressible by what is called the *sine-integral* defined by

$$\mathrm{Si}(\theta) = \int_0^{\theta}\frac{\sin\theta}{\theta}d\theta. \tag{3}$$

Thus

$$\phi(x) = \mathrm{Si}\,k_1(x+1)-\mathrm{Si}\,k_1(x-1), \tag{4}$$

and if the sine-integral were thoroughly known there would be scarcely anything more to do. For moderate values of θ the integral may be calculated from an ascending series which is always convergent. For larger values this series becomes useless; we may then fall back upon a descending series of the semi-convergent class, viz.,

$$\mathrm{Si}(\theta) = \frac{\pi}{2}-\cos\theta\left\{\frac{1}{\theta}-\frac{1.2}{\theta^3}+\frac{1.2.3.4}{\theta^5}-\ldots\right\}$$

$$-\sin\theta\left\{\frac{1}{\theta^2}-\frac{1.2.3}{\theta^4}+\frac{1.2.3.4.5}{\theta^6}-\ldots\right\}. \tag{5}$$

Dr. Glaisher* has given very complete tables extending from $\theta = 0$ to $\theta = 1$, and also from 1 to 5 at intervals of 0·1. Beyond this point he gives the function for integer values of θ from 5 to 15 inclusive, and afterwards only at intervals of 5 for 20, 25, 30, 35, etc. For my purpose these do not suffice, and I have calculated from (5) the values for the missing integers up to $\theta = 60$. The results are recorded in the Table on p. 320. In each case, except those quoted from Glaisher, the last figure is subject to a small error.

For the further calculation, involving merely subtractions, I have selected the special cases $k_1 = 1, 2, 10$. For $k_1 = 1$, we have

$$\phi(x) = \mathrm{Si}(x+1)-\mathrm{Si}(x-1). \tag{6}$$

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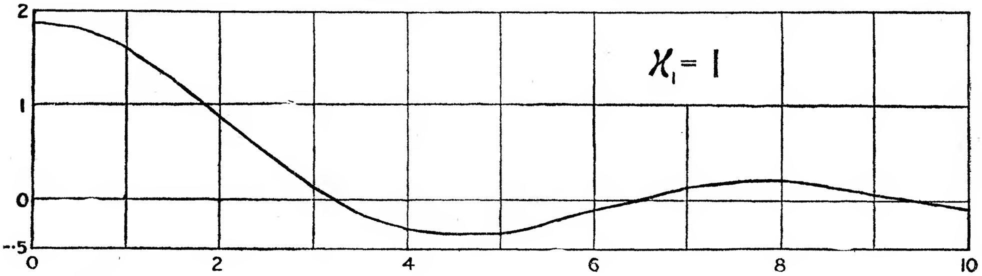
* 'Phil. Trans.,' vol. 160, p. 367 (1870).

θ.	Si (θ).	θ.	Si (θ).	θ.	Si (θ).	θ.	Si (θ).
16	1·63130	28	1·60474	39	1·56334	50	1·55162
17	1·59013	29	1·59731	40	1·58699	51	1·55600
18	1·53662	30	1·56676	41	1·59494	52	1·57357
19	1·51863	31	1·54177	42	1·58083	53	1·58798
20	1·54824	32	1·54424	43	1·55836	54	1·58634
21	1·59490	33	1·57028	44	1·54808	55	1·57072
22	1·61609	34	1·59525	45	1·55871	56	1·55574
23	1·59546	35	1·59692	46	1·57976	57	1·55490
24	1·55474	36	1·57512	47	1·59184	58	1·56845
25	1·53148	37	1·54861	48	1·58445	59	1·58368
26	1·54487	38	1·54549	49	1·56507	60	1·58675
27	1·58029						

In every case $\phi(x)$ is an even function, so that it suffices to consider x positive.

$k_1 = 1$.

x.	$\phi(x)$.	x.	$\phi(x)$.	x.	$\phi(x)$.
0·0	+1·8922	2·5	+0·5084	6·0	−0·0953
0·5	1·8178	3·0	+0·1528	7·0	+0·1495
1·0	1·6054	3·5	−0·1244	8·0	+0·2104
1·5	1·2854	4·0	−0·2987	9·0	+0·0842
2·0	0·9026	5·0	−0·3335	10·0	−0·0867



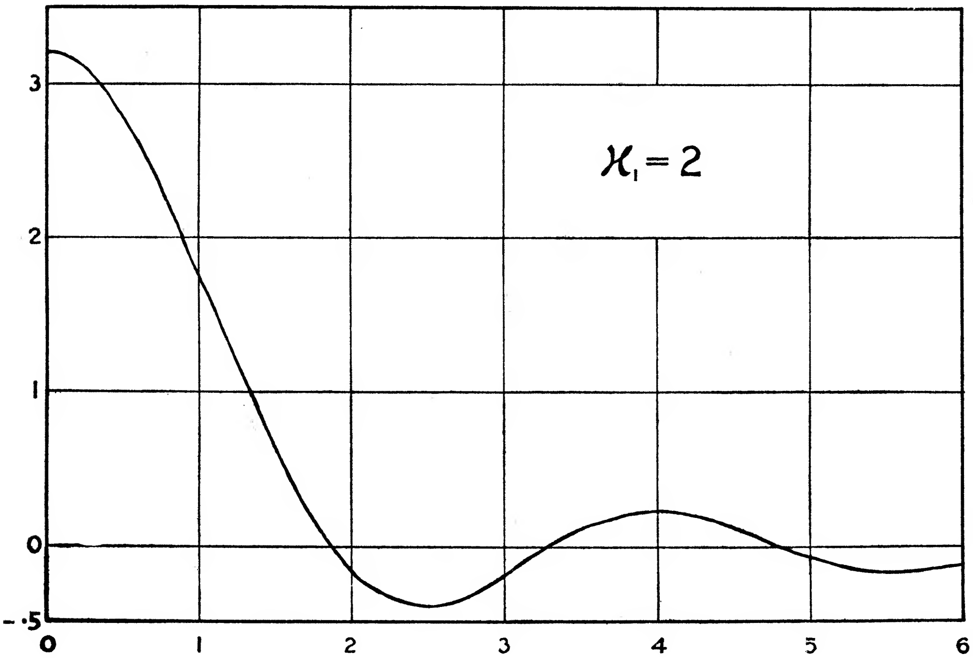
When $k_1 = 2$, $\phi(x) = \mathrm{Si}(2x+2) - \mathrm{Si}(2x-2)$, (7)

and we find

$k_1 = 2$.

x.	$\phi(x)$.	x.	$\phi(x)$.	x.	$\phi(x)$.
0·0	+3·2108	0·9	+1·9929	3·0	−0·1840
0·1	3·1934	1·0	1·7582	3·5	+0·1151
0·2	3·1417	1·1	1·5188	4·0	+0·2337
0·3	3·0566	1·2	1·2794	4·5	+0·1237
0·4	2·9401	1·3	1·0443	5·0	−0·0692
0·5	2·7947	1·4	0·8179	5·5	−0·1657
0·6	2·6235	1·5	+0·6038	6·0	−0·1021
0·7	2·4300	2·0	−0·1807		
0·8	2·2184	2·5	−0·3940		

Both for $k_1 = 1$ and for $k_1 = 2$ all that is required for the above values of $\phi(x)$ is given in Glaisher's tables.

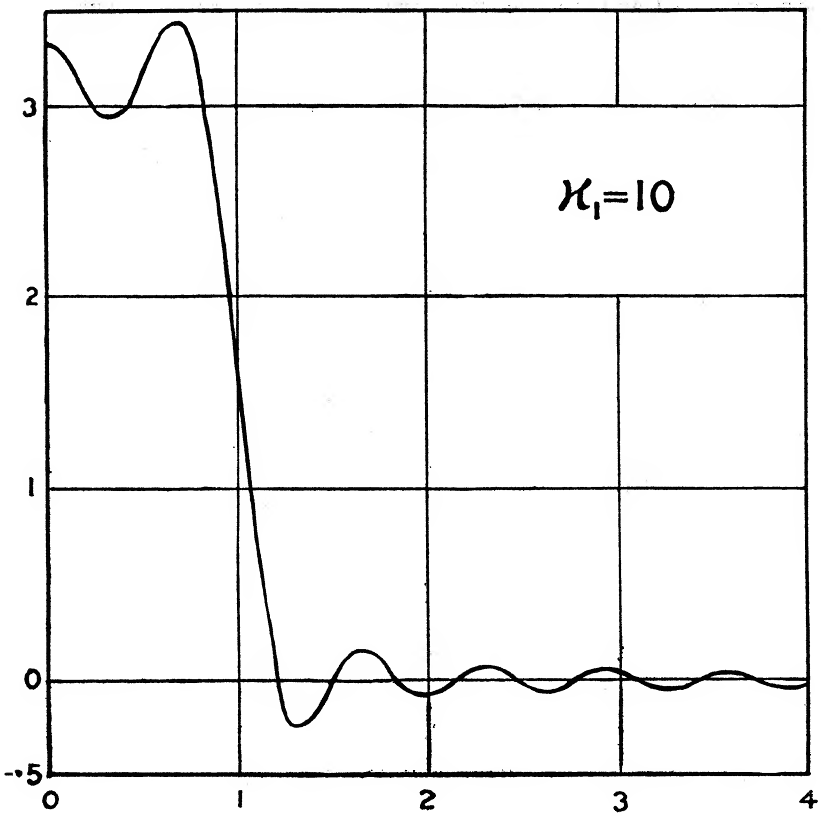


When $k_1 = 10$, $\phi(x) = \mathrm{Si}(10x+10) - \mathrm{Si}(10x-10)$. (8)

We find

$k_1 = 10$.

x.	$\phi(x)$.	x.	$\phi(x)$.	x.	$\phi(x)$.
0·0	+3·3167	1·7	+0·1257	3·4	−0·0067
0·1	3·2433	1·8	+0·0305	3·5	+0·0272
0·2	3·0792	1·9	−0·0677	3·6	+0·0349
0·3	2·9540	2·0	−0·0916	3·7	+0·0115
0·4	2·9809	2·1	−0·0365	3·8	−0·0203
0·5	3·1681	2·2	+0·0393	3·9	−0·0322
0·6	3·3895	2·3	+0·0709	4·0	−0·0151
0·7	3·4388	2·4	+0·0390	4·1	+0·0142
0·8	3·1420	2·5	−0·0213	4·2	+0·0293
0·9	2·4647	2·6	−0·0562	4·3	+0·0178
1·0	1·5482	2·7	−0·0415	4·4	−0·0089
1·1	0·6488	2·8	+0·0089	4·5	−0·0262
1·2	+0·0107	2·9	+0·0447	4·6	−0·0194
1·3	−0·2532	3·0	+0·0387	4·7	+0·0063
1·4	−0·2035	3·1	+0·0000	4·8	+0·0230
1·5	−0·0184	3·2	−0·0353	4·9	+0·0203
1·6	+0·1202	3·3	−0·0371	5·0	−0·0002



The same set of values of Si up to Si (60) would serve also for the calculation of $\phi(x)$ for $k_1 = 20$ and from $x = 0$ to $x = 2$ at intervals of 0·05. It is hardly necessary to set this out in detail.

An inspection of the curves plotted from the above tables shows the approximation towards discontinuity as k_1 increases.

That the curve remains undulatory is a consequence of the sudden stoppage of the integration at $k = k_1$. If we are content with a partial suppression only of the shorter wave-lengths, a much simpler solution is open to us. We have only to introduce into (1) the factor e^{-ak}, where a is positive, and to continue the integration up to $x = \infty$. In place of (2), we have

$$\phi(x) = \int_0^\infty \frac{dk\,e^{-ak}}{k}\{\sin k(x+1) - \sin k(x-1)\} = \tan^{-1}\left(\frac{x+1}{a}\right) - \tan^{-1}\left(\frac{x-1}{a}\right). \tag{9}$$

The discontinuous expression corresponds, of course, to $a = 0$. If a is merely small, the discontinuity is eased off. The following are values of $\phi(x)$, calculated from (9) for $a = 1$, 0·5, 0·05 :—

$a = 1.$

x.	$\phi(x)$.	x.	$\phi(x)$.	x.	$\phi(x)$.
0·0	1·571	2·0	0·464	4·0	0·124
0·5	1·446	2·5	0·309	5·0	0·080
1·0	1·107	3·0	0·219	6·0	0·055
1·5	0·727				

$a = 0{\cdot}5.$

x.	$\phi(x)$.	x.	$\phi(x)$.	x.	$\phi(x)$.
0·00	2·214	1·00	1·326	2·00	0·298
0·25	2·173	1·25	0·888	2·50	0·180
0·50	2·111	1·50	0·588	3·00	0·120
0·75	1·756	1·75	0·408	3·50	0·087

$a = 0{\cdot}05.$

x.	$\phi(x)$.	x.	$\phi(x)$.	x.	$\phi(x)$.
0·00	3·041	0·90	2·652	1·20	0·222
0·20	3·037	0·95	2·331	1·40	0·103
0·40	3·023	1·00	1·546	1·60	0·064
0·60	2·986	1·05	0·761	1·80	0·045
0·80	2·869	1·10	0·440	2·00	0·033

As is evident from the form of (9), $\phi(x)$ falls continuously as x increases, whatever may be the value of a.